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LETTER TO THE EDITOR

On the dynamical group of the charge–monopole system

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Abstract. The massless Poincaré algebra recently discussed is a subalgebra of the dynamical algebra O(4, 2) of the charge-monopole system. Additional generators are given and the interpretation of mass zero representation is elucidated.

The dynamical Poincaré group given recently by Bacry (1981) for the charge monopole system is a subgroup of the more general dynamical conformal group O(4, 2). In addition to charge monopole quantisation, the latter also solves the problem of the energy spectrum and complete degeneracy of the levels (Barut and Bornzin 1971, see also Barut and Raczka 1977). This result is also valid for the more general dyon–dyon system (dyonium). The generators of O(4, 2), in addition to those of the above mentioned Poincaré group, consist of the dilatation operator

$$D = \boldsymbol{r} \cdot \boldsymbol{\pi} - \mathrm{i}; \qquad \boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}, \tag{1}$$

one of the combinations of the Lenz–Runge vector $\boldsymbol{A}$ and $\boldsymbol{M}$, with $\boldsymbol{M} - \boldsymbol{A} = \boldsymbol{r}$ (Bacry's $\boldsymbol{P}$), and

$$\boldsymbol{M} + \boldsymbol{A} = \boldsymbol{r}\pi^2 - 2\boldsymbol{\pi}(\boldsymbol{r} \cdot \boldsymbol{\pi}) + 2\mu(\boldsymbol{r} \times \boldsymbol{\pi}/r) - 2(\mu^2/2r^2)\boldsymbol{r} \tag{2}$$

($\mu = eg$), and one of the combinations of Γ_0, Γ_4 with $\Gamma_0 - \Gamma_4 = r$ (Bacry's H) and

$$\Gamma_0 + \Gamma_4 = r\pi^2 + \mu^2/r. \tag{3}$$

The Casimir operators of the Poincaré subalgebra of O(4, 2) correspond to zero 'mass' and 'helicity' μ. Bacry does further try to give to this realisation a zero-mass interpretation and proposes a position operator for massless particles. However, the coordinates $\boldsymbol{r}$ and $\boldsymbol{p}$ here refer to relative coordinates of two particles, and there is a geometric reason for the occurrence of 'mass zero' Poincaré representations which have nothing to do with zero-mass particles. And there is a systematic way of deriving the above nonlinear realisations of the conformal group (Barut 1980).

In the same way as one derives the usual four-dimensional (in Minkowski Space) nonlinear realisation of the conformal group from a linear six-dimensional representation by restricting the coordinates to lie on a cone $\eta_\mu \eta^\mu - \eta_5 \eta_6 = 0$ in the 6-space (with coordinates η_a, $a = 1 \ldots 6$), one can go one step further, and restrict, in turn, this four-dimensional realisation to one in three dimensions by restricting the coordinates to the light cone $x_\mu x^\mu = 0$ in the Minkowski Space. This is possible, because the light cone is invariant under conformal transformations. In the dual momentum space we similarly have the light cone $p_\mu p^\mu = 0$. These relations allow one to replace, in the usual conformal generators, x^0 by r, or p^0 by $p = \sqrt{\boldsymbol{p}^2}$, and so one obtains the above generators.

The crucial relation $x_\mu x^\mu = 0$ for the relative coordinates can be derived starting from a conformal two-body problem with the relative coordinates $x_{1\mu} - x_{2\mu} = x_\mu$ (Barut and Bornzin 1974) and physically may be connected with the fact that signals travel on the light cone.

It is further important to remark that the dynamical group (O(4, 2), hence its Poincaré subgroup) that is needed for our problem acts in the space of the relative momenta, and not the relative coordinates, showing again that it is different from the space–time Poincaré group defining physical particles.

The 'canonical momenta' do not commute

$$[\pi_i, \pi_j] = \mathrm{i}\mu\varepsilon_{ijk}\hat{r}_k/r^2, \tag{4}$$

because of the contribution of the field momentum.

Finally, all results remain valid for $\mu = 0$ and we get the well known dynamical group of the Coulomb problem. In this case the variables $\boldsymbol{R} = \boldsymbol{r}$ commute so it is again not possible to interpret $\boldsymbol{R}$ as the position operator of a massless particle.

The Casimir operator of O(4, 2) is given by

$$Q^2 = -3(1-\mu^2) \tag{5}$$

because the Casimir operator of the Poincaré subgroup is zero, the right-hand side is also equal to the sum of squares of the remaining five generators $(\boldsymbol{M}+\boldsymbol{A})$, $(\Gamma_0 + \Gamma_\mu)$ and T, as can be checked using the representation relations

$$\{L_{AB}, L_C^A\} = -2(1-\mu^2)g_{BC}; \qquad A, B = 1 \ldots 6$$

for the above realisation of O(4, 2).

References

Bacry H 1981 *J. Phys. A: Math. Gen.* **14** L73

Barut A O 1980 in *Groups, Systems and Many-Body Physics* ed. P Kramer (Vieweg: Verlag) ch VI

Barut A O and Bornzin G 1971 *J. Math. Phys.* **12** 841

—— 1974 *J. Math. Phys.* **15** 1000

Barut A O and Rączka R 1977 *Theory of Group Representations and Applications* (Warsaw: PWN) ch 12–13